Rotation About an Arbitrary Axis Direction

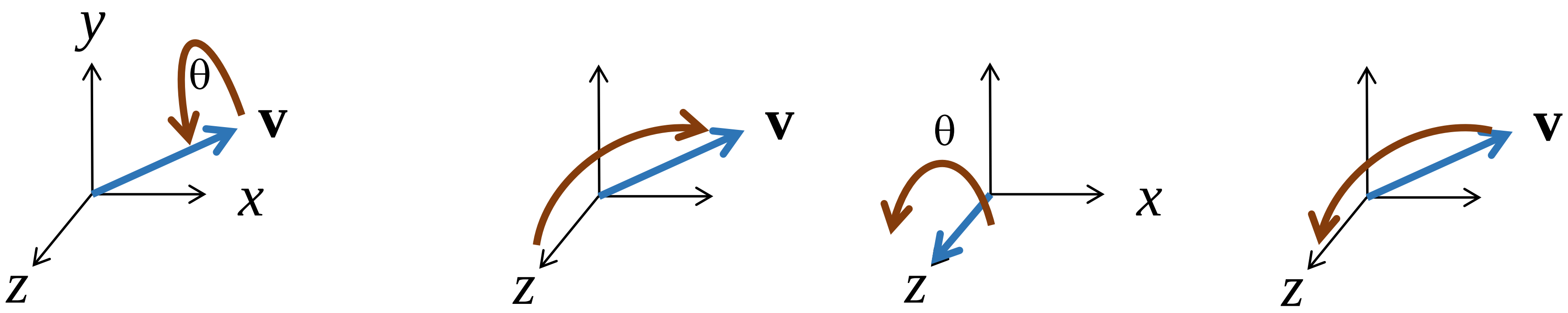
CS418 Computer Graphics

John C. Hart

Arbitrary Axis Rotation

- Find a rotation matrix that rotates by an angle θ about an arbitrary unit direction vector v

$$\mathbf{v} = (x_v, y_v, z_v), \quad x_v^2 + y_v^2 + z_v^2 = 1$$

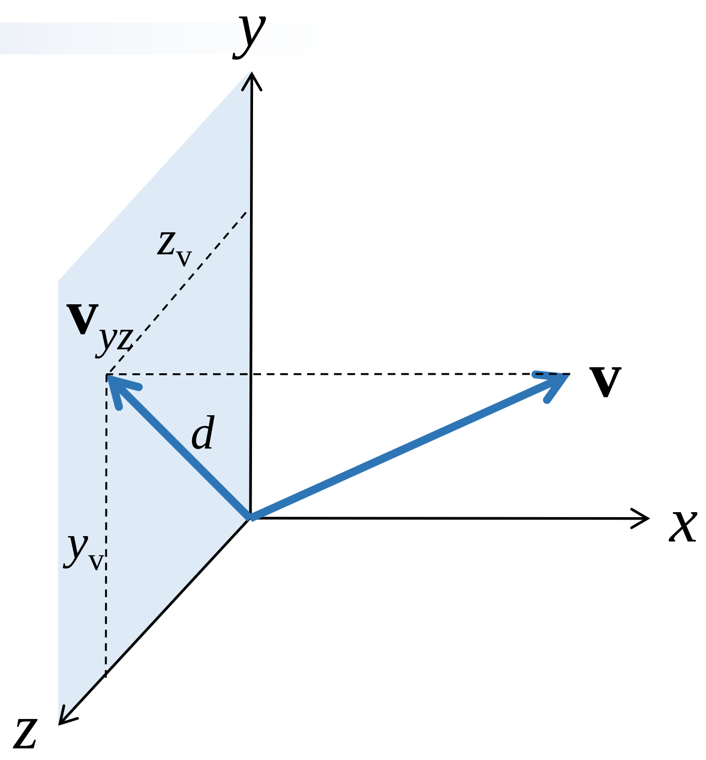


$$\left[\begin{array}{c}\text{Rotate}\\ \text{by }\theta\\ \text{about v}\end{array}\right] = \left[\begin{array}{c}\text{Rotate}\\ \text{z to }\mathbf{v}\end{array}\right]\left[\begin{array}{c}\text{Rotate}\\ \text{by }\theta\\ \text{about z}\end{array}\right]\left[\begin{array}{c}\text{Rotate}\\ \mathbf{v}\text{ to z}\end{array}\right]$$

Rotate v to z

$\mathbf{v} = (x_v, y_v, z_v)$, $x_v^2 + y_v^2 + z_v^2 = 1$

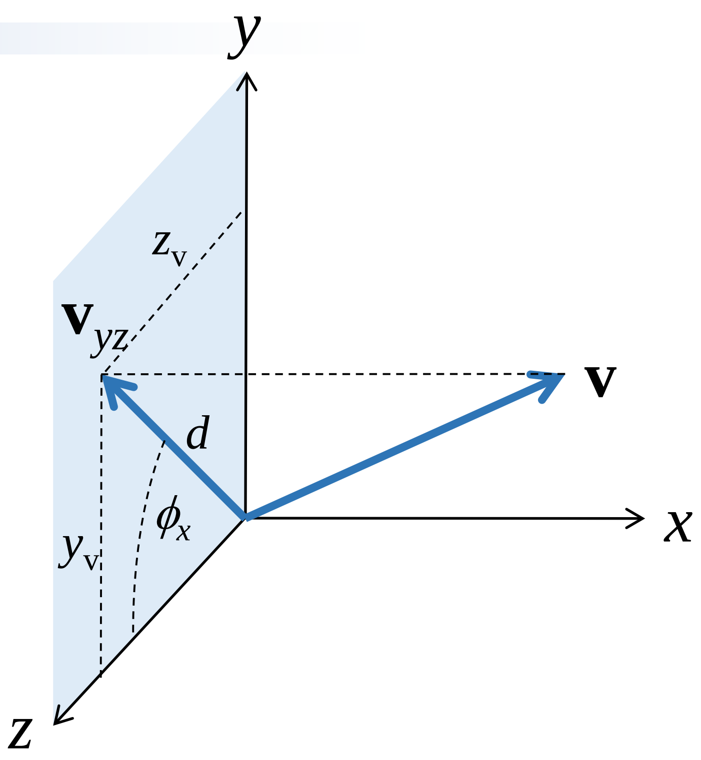
1. Project $\mathbf{v}$ onto the *yz* plane and let $d = \mathrm{sqrt}(y_v^2 + z_v^2)$



Rotate v to z

$\mathbf{v} = (x_v, y_v, z_v)$, $x_v^2 + y_v^2 + z_v^2 = 1$

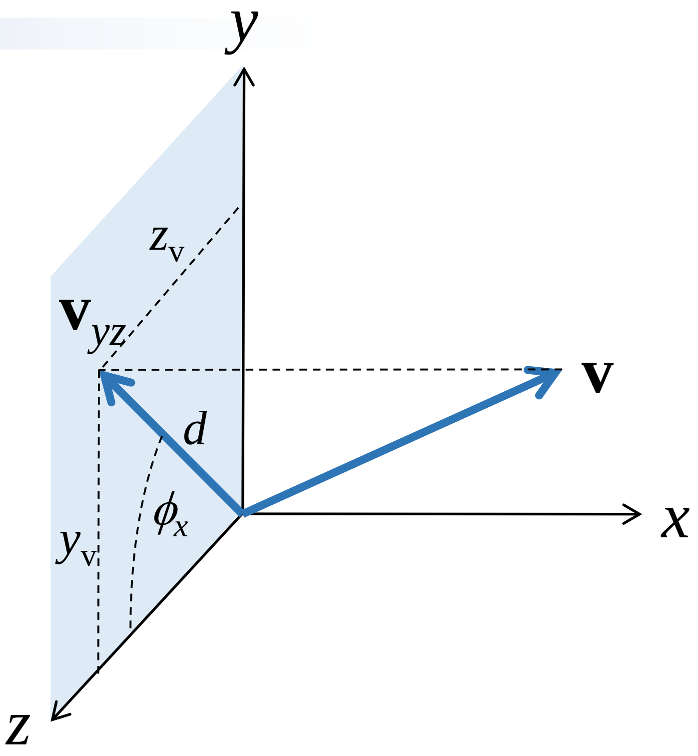
1. Project $\mathbf{v}$ onto the yz plane and let $d = \mathrm{sqrt}(y_v^2 + z_v^2)$
2. Then $\cos\phi_x = z_v/d$ and $\sin\phi_x = y_v/d$



Rotate v to z

$\mathbf{v} = (x_v, y_v, z_v)$, $x_v^2 + y_v^2 + z_v^2 = 1$

1. Project $\mathbf{v}$ onto the yz plane and let $d = \mathrm{sqrt}(y_v^2 + z_v^2)$
2. Then $\cos\phi_x = z_v/d$ and $\sin\phi_x = y_v/d$
3. Rotate $\mathbf{v}$ by ϕ_x about x into the xz plane

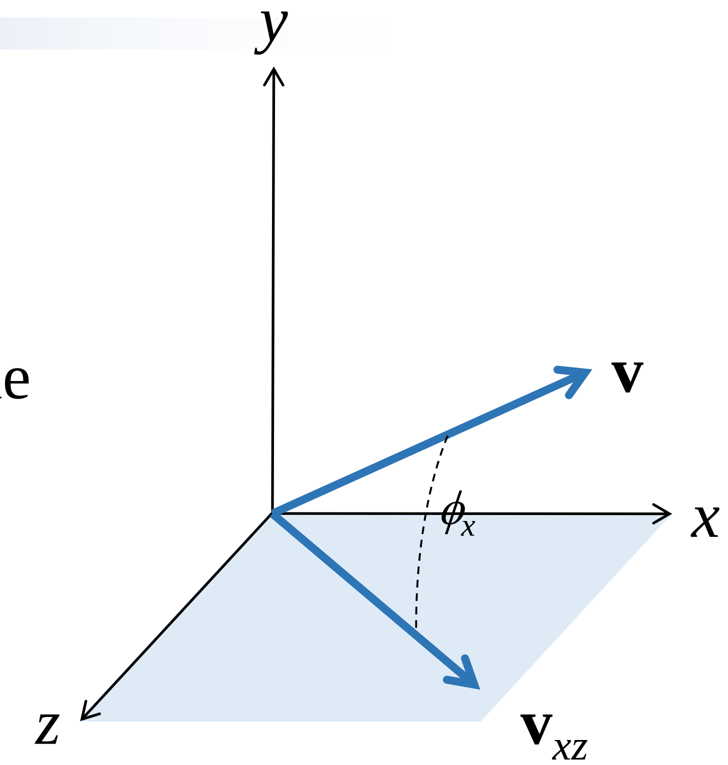


$$\begin{bmatrix} 1 & & & \\ & \frac{z_{\mathbf{v}}}{d} & -\frac{y_{\mathbf{v}}}{d} & \\ & \frac{y_{\mathbf{v}}}{d} & \frac{z_{\mathbf{v}}}{d} & \\ & & & 1 \end{bmatrix}$$

Rotate v to z

$\mathbf{v} = (x_v, y_v, z_v),\ x_v^2 + y_v^2 + z_v^2 = 1$

1. Project $\mathbf{v}$ onto the yz plane and let $d = \mathrm{sqrt}(y_v^2 + z_v^2)$
2. Then $\cos\phi_x = z_v/d$ and $\sin\phi_x = y_v/d$
3. Rotate $\mathbf{v}$ by ϕ_x about x into the xz plane

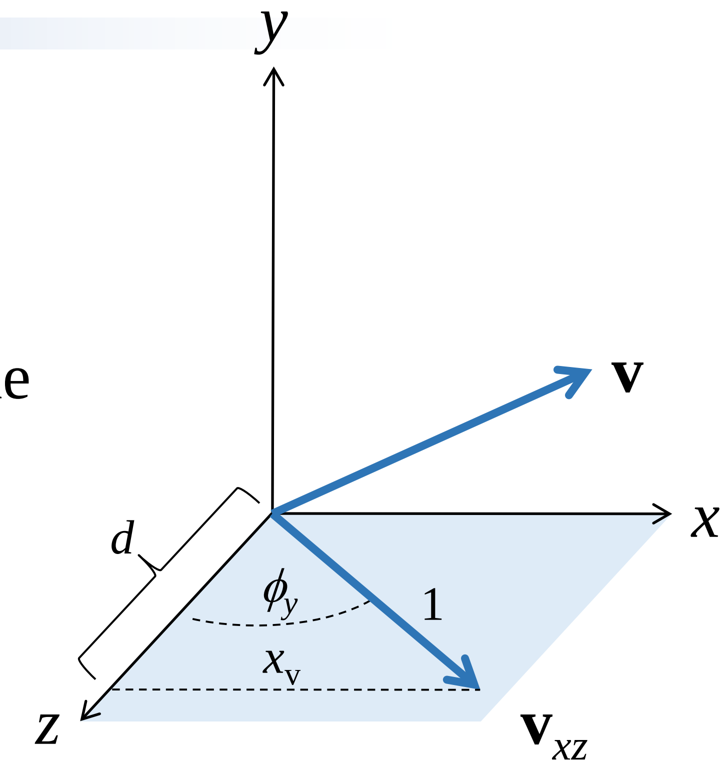


$$\begin{bmatrix} 1 & & & \\ & \frac{z_{\mathbf{v}}}{d} & -\frac{y_{\mathbf{v}}}{d} & \\ & \frac{y_{\mathbf{v}}}{d} & \frac{z_{\mathbf{v}}}{d} & \\ & & & 1 \end{bmatrix}$$

Rotate v to z

$\mathbf{v} = (x_v, y_v, z_v),\ x_v^2 + y_v^2 + z_v^2 = 1$

1. Project $\mathbf{v}$ onto the yz plane and let $d = \mathrm{sqrt}(y_v^2 + z_v^2)$
2. Then $\cos\phi_x = z_v/d$ and $\sin\phi_x = y_v/d$
3. Rotate $\mathbf{v}$ by ϕ_x about x into the xz plane
4. Then $\cos\phi_y = d$ and $\sin\phi_y = x_v$

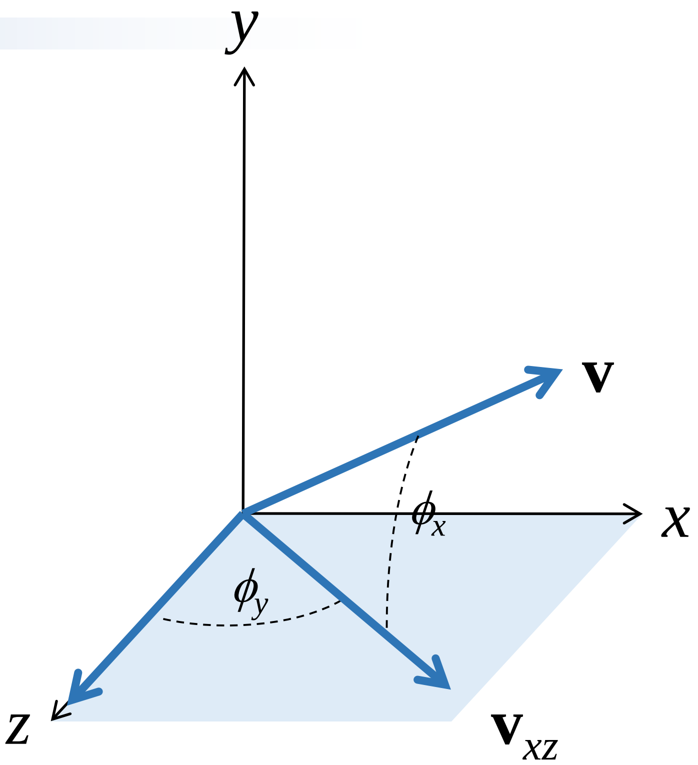


$$\begin{bmatrix} 1 & & & \\ & \frac{z_{\mathbf{v}}}{d} & -\frac{y_{\mathbf{v}}}{d} & \\ & \frac{y_{\mathbf{v}}}{d} & \frac{z_{\mathbf{v}}}{d} & \\ & & & 1 \end{bmatrix}$$

Rotate v to z

$\mathbf{v} = (x_\mathrm{v}, y_\mathrm{v}, z_\mathrm{v}),\ x_\mathrm{v}^2 + y_\mathrm{v}^2 + z_\mathrm{v}^2 = 1$

1. Project **v** onto the *yz* plane and let $d = \mathrm{sqrt}(y_\mathrm{v}^2 + z_\mathrm{v}^2)$
2. Then $\cos\phi_x = z_\mathrm{v}/d$ and $\sin\phi_x = y_\mathrm{v}/d$
3. Rotate **v** by ϕ_x about x into the xz plane
4. Then $\cos\phi_y = d$ and $\sin\phi_y = x_\mathrm{v}$
5. Rotate $\mathbf{v}_\mathrm{xz}$ by ϕ_y about y into the z axis



$$\begin{bmatrix} d & & -x_\mathbf{v} & \\ & 1 & & \\ x_\mathbf{v} & & d & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & \frac{z_\mathbf{v}}{d} & -\frac{y_\mathbf{v}}{d} & \\ & \frac{y_\mathbf{v}}{d} & \frac{z_\mathbf{v}}{d} & \\ & & & 1 \end{bmatrix}$$

Rotate θ about **v**

- Let $R_{\mathbf{v}}(\theta)$ be the rotation matrix for rotation by θ about arbitrary axis direction **v**
- Recall $(R_x\ R_y)$ is the matrix (product) that rotates direction **v** to z axis
- Then

$$R_{\mathbf{v}}(\theta) = (R_y\ R_x)^{-1}\ R_z(\theta)\ (R_y\ R_x)$$
$$= R_x^{-1}\ R_y^{-1}\ R_z(\theta)\ R_y\ R_x$$
$$= R_x^{T}\ R_y^{T}\ R_z(\theta)\ R_y\ R_x$$

(since the inverse of a rotation matrix is the transpose of the rotation matrix)

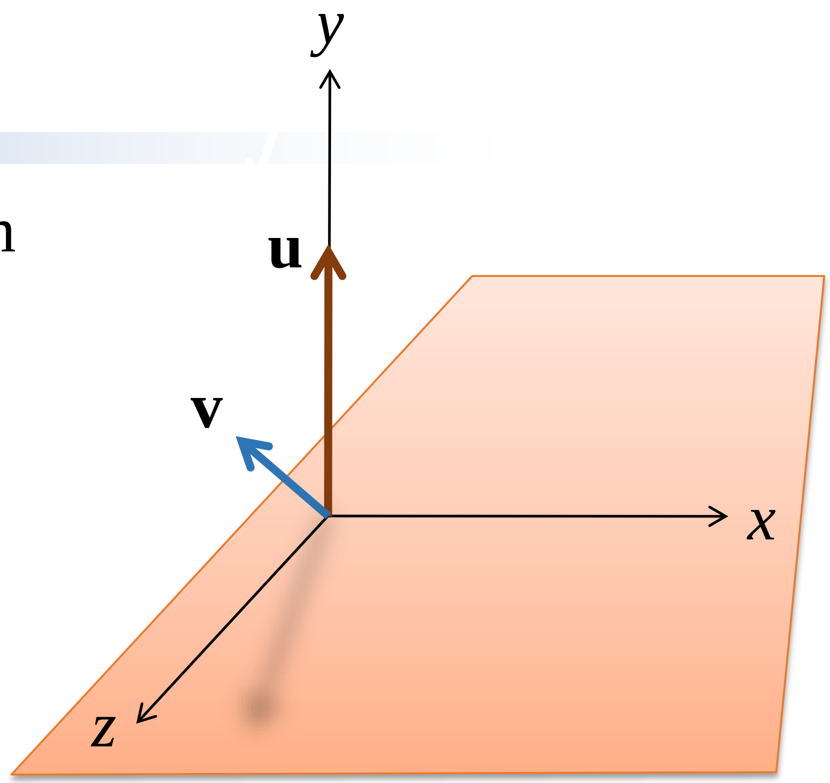
$$R_x = \begin{bmatrix} 1 & & & \\ & \frac{z_v}{d} & -\frac{y_v}{d} & \\ & \frac{y_v}{d} & \frac{z_v}{d} & \\ & & & 1 \end{bmatrix}$$

$$R_y = \begin{bmatrix} d & & x_v & \\ & 1 & & \\ x_v & & d & \\ & & & 1 \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & & \\ \sin\theta & \cos\theta & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

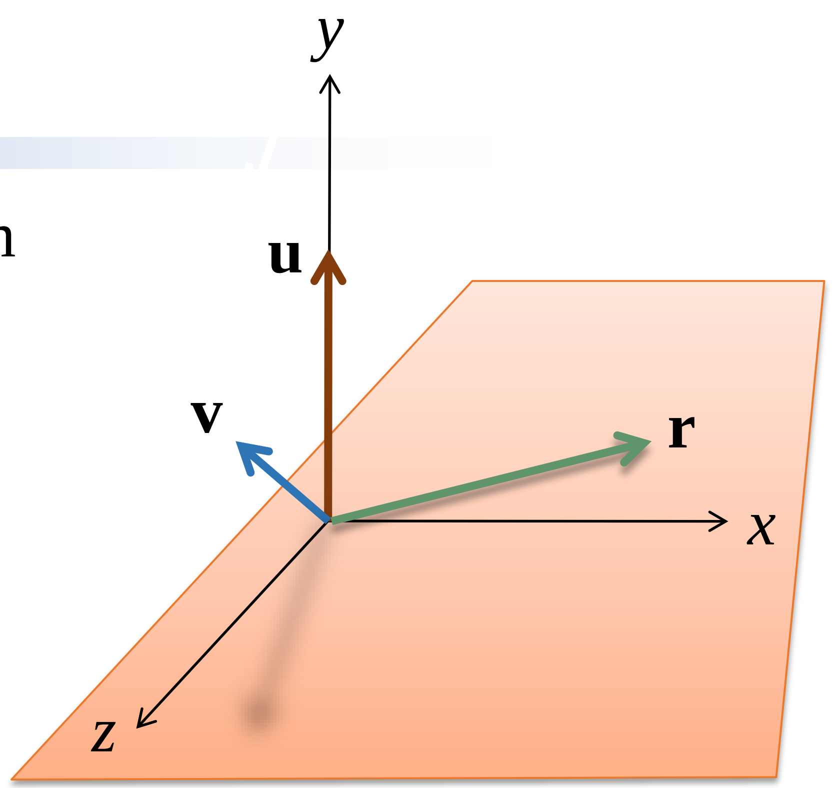
Easier Way

- Find an orthonormal vector system



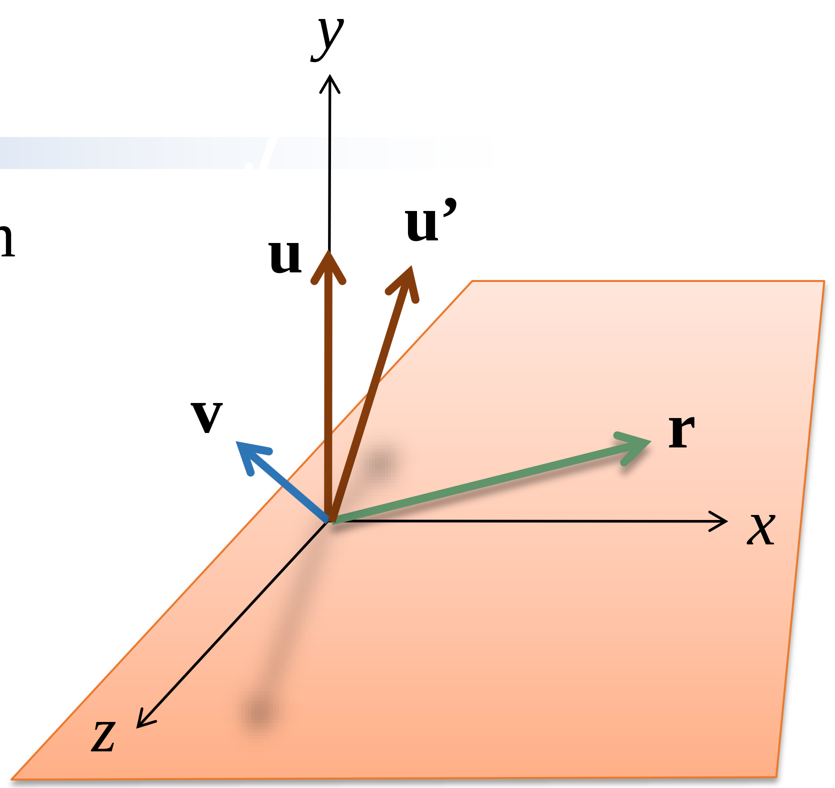
Easier Way

- Find an orthonormal vector system
  - Let $\mathbf{r} = \mathbf{u} \times \mathbf{v}/||\mathbf{u}\times\mathbf{v}||$



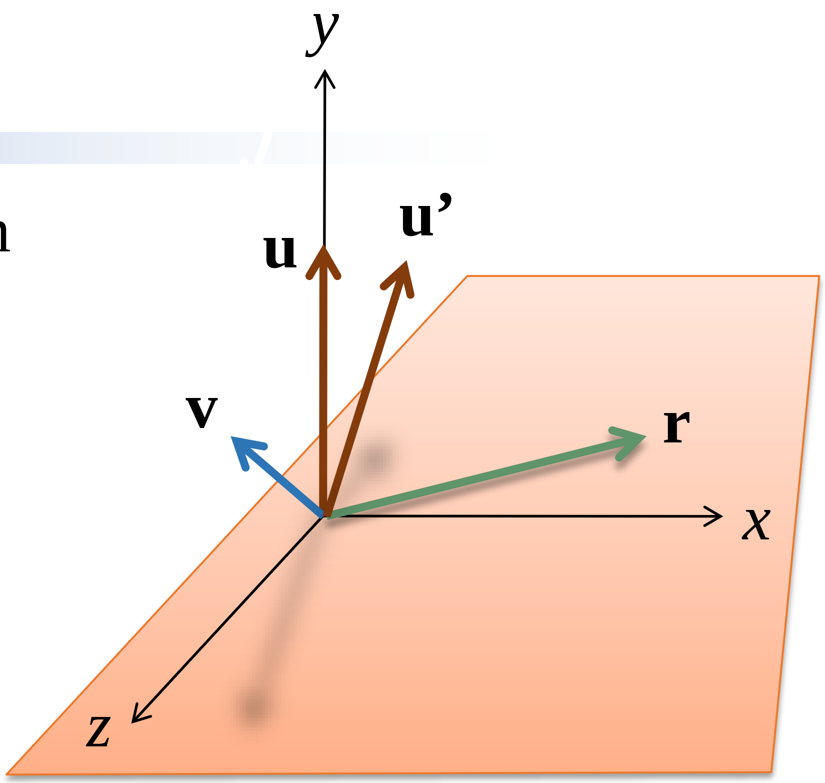
Easier Way

- Find an orthonormal vector system
  - Let $\mathbf{r} = \mathbf{u} \times \mathbf{v}/||\mathbf{u}\times\mathbf{v}||$
  - Let $\mathbf{u}' = \mathbf{v} \times \mathbf{r}$



Easier Way

- Find an orthonormal vector system
  - Let **r** = **u** × **v**/||**u**×**v**||
  - Let **u’** = **v** × **r**
- Find a rotation from <**r**,**u’**,**v**> → <**x**,**y**,**z**>

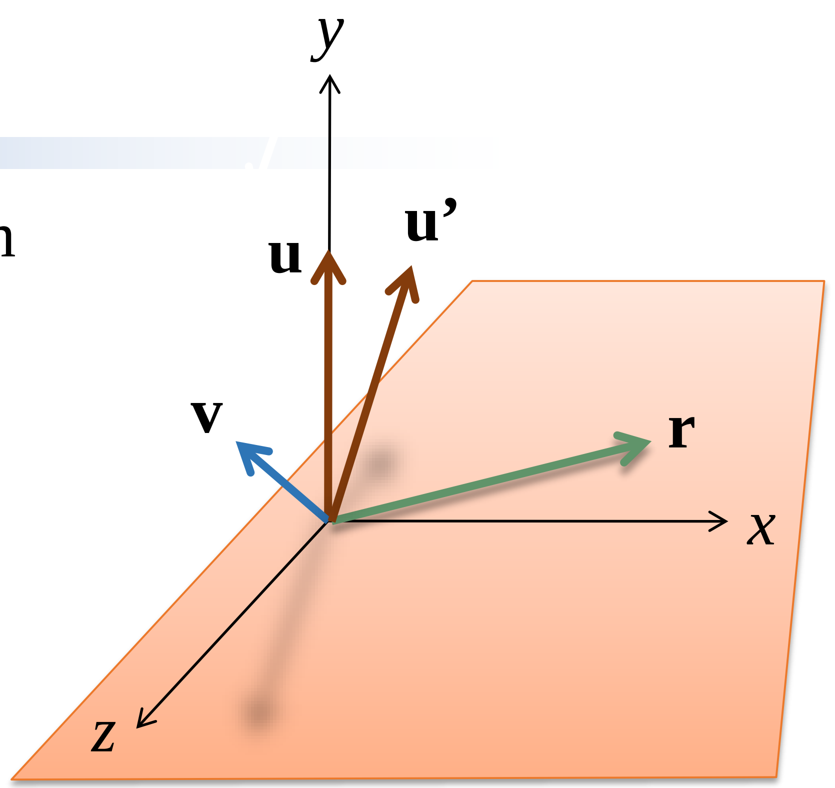


$$\begin{bmatrix} r_x & u'_x & v_x & \\ r_y & u'_y & v_y & \\ r_z & u'_z & v_z & \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} r_x \\ r_y \\ r_z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} r_x & r_y & r_z & \\ u'_x & u'_y & u'_z & \\ v_x & v_y & v_z & \\ & & & 1 \end{bmatrix}\begin{bmatrix} r_x \\ r_y \\ r_z \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Easier Way

- Find an orthonormal vector system
  - Let **r** = **u** × **v**/||**u**×**v**||
  - Let **u’** = **v** × **r**
- Find a rotation
  from <**r**,**u’**,**v**> → <**x**,**y**,**z**>



$$\begin{bmatrix} r_x & u'_x & v_x & \\ r_y & u'_y & v_y & \\ r_z & u'_z & v_z & \\ & & & 1\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta & & \\ \sin\theta & \cos\theta & & \\ & & 1 & \vdots \\ & & & 1\end{bmatrix}\begin{bmatrix} r_x & r_y & r_z & \\ u'_x & u'_y & u'_z & \\ v_x & v_y & v_z & \\ & & & 1\end{bmatrix}$$